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Topics in complex analysis
Fall 2025
Sheet 1

Solution for 'Topics in complex analysis'

(10/09/2025)

H 1.1 (Does it exist or not?)
Give an example of the following objects or prove that they cannot exist.

(i) A holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $f'(z) = \frac{1}{z}$.

(ii) A holomorphic function $f : \mathbb{C} \setminus \{\mathrm{Re}\,(z) \le 0\} \to \mathbb{C}$ such that $f'(z) = \frac{1}{z}$.

(iii) A domain $D \subset \mathbb{C}$ and a non-constant holomorphic function $f : D \to \mathbb{C}$ such that $|f|$ is constant.

(iv) A non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(n) = 0$ for all $n \in \mathbb{N}$.

(v) A non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $f(\frac{1}{n} + i) = 0$ for all $n \in \mathbb{N}$.

(vi) A non-constant holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that is bounded.

Solution H 1.1:

(i) No. Otherwise, by the fundamental theorem of calculus and the definition of complex path integrals we would obtain

$$0 = f(1) - f(1) = \int_{\partial B_1(0)} f'(z)\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{e^{it}} e^{it} i\,\mathrm{d}t = 2\pi i.$$

(ii) Yes. On star-shaped domains, every holomorphic function possesses a primitive. One example in this case is the complex logarithm $f(z) = \log(z)$ (more precisely its principal value with imaginary part in $(-\pi, \pi]$).

(iii) No. Such a function would satisfy $f(D) \subset \partial B_r(0)$ for some $r \ge 0$. In particular, the set $f(D)$ is not open, which contradicts the open mapping theorem.

(iv) Yes. Consider the function $f(z) = \sin(\pi z)$.

(v) No. Since the sequence $\frac{1}{n} + i$ has the accumulation point $i \in \mathbb{C}$, the identity theorem yields that $f(z) = 0$ for all $z \in \mathbb{C}$.

(vi) No. Since the point z_0 is an isolated singularity, Riemann's lemma implies that z_0 would be a removable singularity. Then the extended function would violate Liouville's theorem.

□

H 1.2 (On the image of entire functions)
Let $f : \mathbb{C} \to \mathbb{C}$ be an non-constant entire function (i.e. f is holomorphic on the whole complex plane). Show that its image $f(\mathbb{C})$ is dense in $\mathbb{C}$.
Hint: Argue by contradiction and use Liouville's theorem for a suitable function.

Solution H 1.2:
Assume by contradiction that there exists $z_0 \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - z_0| > \delta$ for all $z \in \mathbb{C}$. (Check carefully that this is exactly the contradiction of the statement!) Then the function $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) = \frac{1}{f(z) - z_0}$$

satisfies $|g(z)| \leq \frac{1}{\delta}$. Hence by Liouville's theorem g has to be constant, which contradicts the assumption that f is non-constant. □

H 1.3 (The Arzelà-Ascoli theorem on $\mathbb{C}$)
Let $K \subset \mathbb{C}$ be a compact set and let $f_n : K \to \mathbb{C}$ be an equibounded sequence of functions, i.e., there exists a constant $M > 0$ such that

$$\sup_{z \in K} |f_n(z)| \leq M \quad \forall n \in \mathbb{N}.$$

a) Let $S \subset K$ be a countable set. Show that there exists a subsequence f_{n_k} such that $f_{n_k}(z)$ converges to some value $f_z \in \mathbb{C}$ for all $z \in S$.

b) Now assume in addition that the sequence $f_n : K \to \mathbb{C}$ is equicontinuous, i.e., for all $\varepsilon > 0$ and $z \in K$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ we have the implication

$$|y - x| < \delta \quad \Rightarrow \quad |f_n(y) - f_n(x)| < \varepsilon.$$

Show that if S is dense in K, then $f_{n_k}(z)$ converges to some $f_z \in \mathbb{C}$ for all $z \in K$, where f_{n_k} denotes the subsequence from item a).

c) Under the assumptions of b), define the function $f : K \to \mathbb{C}$ by $f(z) := f_z$. Show that $f_{n_k} \to f$ uniformly on K and conclude that f is continuous.

Solution H 1.3:
a) Let us write $S = \{z_1, z_2, z_3, \dots\}$. Since the sequence $\{f_n(z_1)\}_{n \in \mathbb{N}}$ is bounded, we may apply the Bolzano-Weierstrass theorem in order to extract a subsequence $\{n_{k,1}\}_{k \in \mathbb{N}}$ such that $f_{n_{k,1}}(z_1)$ converges to some value $f_{z_1} \in \mathbb{C}$. In the next step we note that the sequence $\{f_{n_{k,1}}(z_2)\}_{k \in \mathbb{N}}$ is again bounded, so that by the same reasoning we find another subsequence $\{n_{k,2}\}_{k \in \mathbb{N}}$ of the previous subsequence, such that $\{f_{n_{k,2}}(z_2)\}_{k \in \mathbb{N}}$ converges to some value $f_{z_2} \in \mathbb{C}$. In the j^{th} step we choose a subsequence $\{n_{k,j}\}_{k \in \mathbb{N}}$ of all previous subsequences, such that $\{f_{n_{k,j}}(z_j)\}_{k \in \mathbb{N}}$ converges to some value $f_{z_j} \in \mathbb{C}$.
It is tempting to finish the argument by induction, however it is not clear that any sequence remains after this infinite procedure! To justify this, we use the so-called *diagonal argument.* For $k \in \mathbb{N}$ we set $n_k := n_{k,k}$. Then the sequence $f_{n_k}(z_j)$ converges to f_{z_j} for all $j \in \mathbb{N}$, since apart from finitely many terms (the ones with $k < j$), the sequence n_k is a subsequence of $\{n_{k,j}\}_{k \in \mathbb{N}}$.
Remark: This type of diagonal argument is attributed to Cantor, and has applications in many fields of mathematics.

b) We show that $\{f_{n_k}(z)\}_{k \in \mathbb{N}}$ is a Cauchy sequence for all $z \in K$. To reduce notation, we suppress the subscript k. For $z \in K$ and $\varepsilon > 0$ we first choose $z^* \in S$ such that $|z - z^*| < \delta_{\varepsilon,z}$, where $\delta_{\varepsilon,z}$ satisfies the condition

$$|y - z| < \delta_{\varepsilon,z} \quad \Rightarrow \quad |f_n(y) - f_n(z)| < \frac{\varepsilon}{3} \qquad \forall n \in \mathbb{N}. \tag{1}$$

It is possible to find such a z^* due to the density of S in K. For $m \geq n$ we then have

$$|f_m(z) - f_n(z)| \leq \underbrace{|f_m(z) - f_m(z^*)|}_{<\varepsilon/3} + |f_m(z^*) - f_n(z^*)| + \underbrace{|f_n(z^*) - f_n(z)|}_{<\varepsilon/3}.$$

Since $z^* \in S$, part a) implies that there exists $n = n_\varepsilon \in \mathbb{N}$ such that for all $m \geq n$ we have $|f_m(z^*) - f_n(z^*)| < \frac{\varepsilon}{3}$. Then for all $m \geq n \geq n_\varepsilon$ we conclude that

$$|f_m(z) - f_n(z)| < \varepsilon.$$

Hence $\{f_n(z)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, which implies the desired statement.

c) Given $\varepsilon > 0$ and $z \in K$, we choose $\delta_{\varepsilon,z} > 0$ satisfying the implication (1) above. Then the family of discs $\{B_{\delta_{\varepsilon,z}}(z)\}_{z\in K}$ forms an open cover of the compact set K. By the (topological) definition of compactness, there exists a finite sub-family $\{B_{\delta_{\varepsilon,z_i}}\}_{i=1}^N$ with $z_i \in K$ that still covers K. Thus for any $z \in K$ we find z_i such that $|z - z_i| < \delta_{\varepsilon,z_i}$. Since the $\{z_i\}$ are only finitely many, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ it holds that

$$|f(z_i) - f_n(z_i)| < \frac{\varepsilon}{3}.$$

Moreover, observe that (1) also holds for the limit function f since we can pass to the limit in the estimate. Consequently, for $n \geq n_\varepsilon$ we deduce that for all $z \in K$ we have

$$|f(z) - f_n(z)| \leq \underbrace{|f(z) - f(z_i)|}_{\varepsilon/3} + \underbrace{|f(z_i) - f_n(z_i)|}_{<\varepsilon/3} + \underbrace{|f_n(z_i) - f_n(z)|}_{<\varepsilon/3} < \varepsilon,$$

which shows the uniform convergence of f_n to f. Since uniform convergence preserves continuity (see Analysis 2 or pass to the limit in (1)), this concludes the proof. $\square$